Slope of angle bisectors of rhombus

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Question 1

Given a rhombus ABCD (a parallelogram with all sides equal)

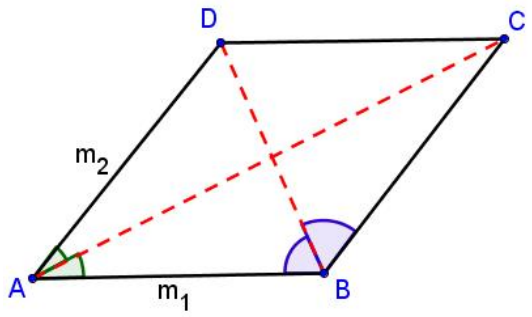
and m_1 = slope of AB or DC

m_2 = slope of AD or BC

prove that the slope of the diagonals are given by

$$m = \frac{a}{b \pm \sqrt{a^2+b^2}}$$

where $a = m_1 + m_2,\ b = 1 - m_1 m_2$.



Since the diagonals bisect the angles at the vertices of the rhombus, **Question 1** is the same as the following question:

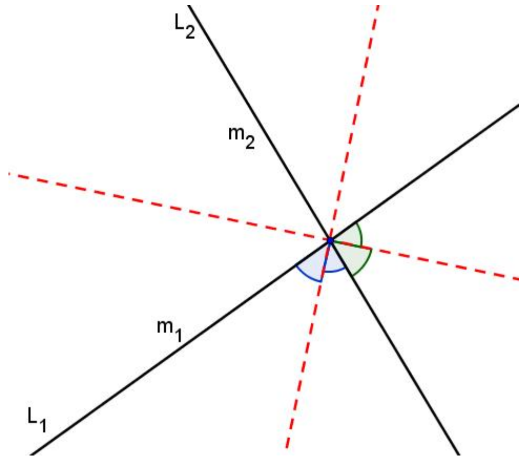
Question 2

Given two lines L_1 and L_2 with slopes m_1, m_2 respectively.

prove that the slope of the angle bisectors are given by

$$m = \frac{a}{b \pm \sqrt{a^2+b^2}}$$

where $a = m_1 + m_2,\ b = 1 - m_1 m_2$.



Both **Question 1** and **Question 2** are difficult to prove,

we need to prove the following identities first :

Identities

(1) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$, $xy < 1$ …. **(1A)**

$\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\frac{x+y}{1-xy}$, $x > 0, xy > 1$ …. **(1B)**

$\tan^{-1}x + \tan^{-1}y = -\pi + \tan^{-1}\frac{x+y}{1-xy}$, $x < 0, xy > 1$ …. **(1C)**

(2) $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$, $|x| < 1$ …. **(2A)**

$2\tan^{-1}x = \pi + \tan^{-1}\frac{2x}{1-x^2}$, $|x| > 1$ …. **(2B)**

$2\tan^{-1}x = -\pi + \tan^{-1}\frac{2x}{1-x^2}$, $x < -1$ …. **(2C)**

(3) $\tan^{-1}x + \tan^{-1}y = 2\tan^{-1}\frac{a}{b+\sqrt{a^2+b^2}}$, where $a = x + y,\ b = 1 - xy$. …. **(3)**

Proof

(1) We only show **(1A)**,

$$\tan(\tan^{-1}x + \tan^{-1}y) = \frac{\tan(\tan^{-1}x) + \tan(\tan^{-1}y)}{1 - \tan(\tan^{-1}x)\tan(\tan^{-1}y)} = \frac{x+y}{1-xy}$$. Result follows.

(2) We only show **(2A)**,

$$\tan(2\tan^{-1}x) = \frac{2\tan(\tan^{-1}x)}{1-[\tan(\tan^{-1}x)]^2} = \frac{2x}{1-x^2}$$. Result follows.

(3) We also have different cases for arctan formula, we only use 1A and 2A here,

By **(1)**, L.H.S. = $\tan^{-1}\frac{x+y}{1-xy} = \tan^{-1}\frac{a}{b}$

By **(2)**, R.H.S. $= 2\tan^{-1}\frac{a}{b+\sqrt{a^2+b^2}} = \tan^{-1}\frac{2\left(\frac{a}{b+\sqrt{a^2+b^2}}\right)}{1-\left(\frac{a}{b+\sqrt{a^2+b^2}}\right)^2} = \tan^{-1}\frac{2a\left(b+\sqrt{a^2+b^2}\right)}{\left(b+\sqrt{a^2+b^2}\right)^2-a^2}$

$= \tan^{-1}\frac{2a\left(b+\sqrt{a^2+b^2}\right)}{b^2+2b\sqrt{a^2+b^2}+a^2+b^2-a^2} = \tan^{-1}\frac{2a\left(b+\sqrt{a^2+b^2}\right)}{2b^2+2b\sqrt{a^2+b^2}} = \tan^{-1}\frac{2a\left(b+\sqrt{a^2+b^2}\right)}{2b\left(b+\sqrt{a^2+b^2}\right)} = \tan^{-1}\frac{a}{b}$

∴ L.H.S. = R.H.S.

Now we come back with the main proof of **Question 2**. (**Question 1** is the same.)

Given two lines L_1 and L_2 with slopes m_1 , m_2 respectively.

Let α and β be the angles of inclination of L_1 and L_2 .

Let γ and δ be the angles of inclination of the bisectors of L_1 and L_2 as in the diagram.

Note that we have various sizes of the angles of inclination, but the proof is the similar except that they may differ by $\pm\pi$ (or $\pm 180°$) by using different formulas in 1A, 1B, 1C and 2A, 2B, 2C.

Check that :

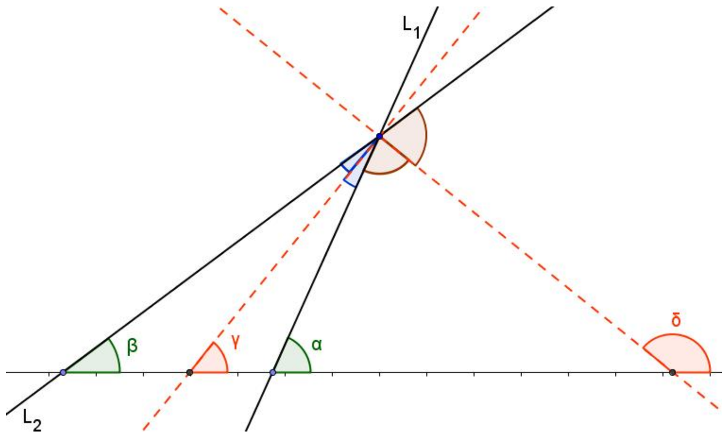
$\gamma = \frac{1}{2}(\alpha+\beta)$

$= \frac{1}{2}(\tan^{-1}m_1 + \tan^{-1}m_2)$

$= \frac{1}{2}\left(2\tan^{-1}\frac{a}{b+\sqrt{a^2+b^2}}\right)$, by **(3)**

$= \tan^{-1}\frac{a}{b+\sqrt{a^2+b^2}}$

where $a = m_1 + m_2,\ b = 1 - m_1 m_2$.



Therefore the slope of one of the angle bisectors is given by

$$m = \tan\gamma = \frac{a}{b+\sqrt{a^2+b^2}} \quad \text{where} \quad a = m_1 + m_2,\ b = 1 - m_1 m_2 .$$

The proof of the slope of the other bisector

$$m = \tan\delta = \frac{a}{b-\sqrt{a^2+b^2}} \quad \text{where} \quad a = m_1 + m_2,\ b = 1 - m_1 m_2 .$$

is left to the reader.

References:

With thanks from Gregory V. Akulov : published in 1999 in Kyiv, Ukraine, in 2009 in Manitoba, Canada, and in 2010 in Alberta, Canada

http://mathcentral.uregina.ca/RR/database/RR.09.10/akulov1.html

http://mathcentral.uregina.ca/RR/database/RR.09.11/akulov1.html